Journal of Statistical Physics, Vol. 96, Nos. 5/6, 1999

Two-Dimensional Potts Model and Annular Partitions

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Received March 22, 1999

Using the random cluster expansion, correlations of the Potts model on a graph can be expressed as sums over partitions of the vertices where the spins are fixed. For a planar graph, only certain partitions can occur in these sums. For example, when all fixed spins lie on the boundary of one face, only noncrossing partitions contribute. In this paper we examine the partitions which occur when fixed spins lie on the boundaries of two disjoint faces. We call these the annular partitions, and we establish some of their basic properties. In particular we demonstrate a partial duality for these partitions, and show some implications for correlations of the Potts model.

KEY WORDS: Potts model; partitions; poset; duality.

1. INTRODUCTION

This paper explores the relation between the 2D Potts model and some classes of partitions. We will assume some familiarity with the Potts model (for a general review see ref. 1), whereas most of our discussion of partitions will be self-contained (see ref. 2 for a good introduction).

Recall that a partition of a set is a collection of disjoint subsets, called blocks, whose union is the entire set. To see how partitions arise for the Potts model on a graph, let U be some subset of vertices of the graph. The random cluster expansion[(3)] of the partition function generates a sum over spanning subgraphs, each with many connected components. For any such subgraph, we say that two of the vertices in U are in the same block if and only if they belong to the same connected component. Then each subgraph generates a partition of U. So the random cluster expansion can be rewritten

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0022-4715/99/0900-1071$16.00/0

as a sum over partitions of U, followed by a sum over all spanning subgraphs which induce that partition on U.

This observation acquires some significance when one considers correlations of the Potts model, with spins being fixed on U. The values of the fixed spins also define a partition Y on U, where two vertices are in the same block of Y if and only if they have the same fixed spin value. Then the only terms which contribute in the random cluster expansion are those whose partition of U is a refinement of Y (see Section 2.2 for the definition of refinement).

Furthermore, when G is a connected planar graph, there are strong restrictions imposed by the graph topology. The most well-known case is where the fixed spins all reside on the boundary of one face, and in this case only *noncrossing* partitions can occur.[4] Recall that a partition of points on the boundary of a disk is *noncrossing* (NC) if the points can be connected by arcs through the interior of the disk, in such a way that (a) each block is path-connected along the arcs, and (b) arcs of different blocks do not intersect. The NC partitions of $\{1, 2,..., n\}$ form a partially ordered set (poset), and this poset has been extensively studied in combinatorics.[5–9] It is not hard to see that only NC partitions contribute when G is planar; indeed, the random cluster expansion provides the connecting arcs for each block (fattened out into subgraphs now). So the random cluster expansion for this model can be re-written as a sum over NC partitions of the vertices with fixed spins. This fact has been exploited in both numerical and analytical studies of the 2D Potts model.[4, 10–12]

Recently Reiner introduced an extended notion of NC partitions,[13] using as motivation the identification of partitions of $\{1, 2,..., n\}$ with the intersection lattice for hyperplane arrangements corresponding to the root system of type A_{n-1}. In keeping with his notation we will write $\mathrm{NC}^{\mathrm{A}}(n)$ for the noncrossing partitions of $\{1, 2,..., n\}$. Other root systems produce intersection lattices which can also be identified with classes of partitions, and Reiner uses a graphical representation of these partitions to define his extended notion of NC partition for a root system.

In this paper we show how consideration of the 2D Potts model leads naturally to another interesting extension of NC partitions, which we call the *annular* partitions. These are found by applying the random cluster expansion to correlations on a planar graph where spins are fixed on the boundaries of *two* disjoint faces. In essence, we fix points on the two boundary components of an annulus, and then consider all the partitions whose blocks can be connected by arcs through the interior of the annulus, in such a way that arcs of different blocks do not intersect. To be specific, suppose there are n points on one boundary component and m points on the other. Then we will write $\mathrm{NC}_2^{\mathrm{A}}(n, m)$ for the class of annular partitions of this set of $n+m$ points. Note that $\mathrm{NC}_2^{\mathrm{A}}(n,m)$ and $\mathrm{NC}_2^{\mathrm{A}}(m,n)$ are naturally isomorphic. Also we will demonstrate the inclusion

$$\mathrm{NC}^{\mathrm{A}}(n)\times\mathrm{NC}^{\mathrm{A}}(m)\subset\mathrm{NC}_2^{\mathrm{A}}(n,m) \tag{1}$$

Since the annulus can be viewed as a cylinder, annular partitions also occur in the random cluster expansion of the Potts model on the cylinder, where spins are fixed at both ends.

In the succeeding sections we will classify the annular partitions, and show how they can be constructed using NC partitions on each boundary component. We also derive a duality relation, analogous to the well-known self-duality of NC partitions,[8, 4] and use it to derive some duality relations for the Potts model correlations.

The paper is organized as follows: in Section 2 we review partitions and their relation to correlations of the Potts model. In Section 3 we show how NC partitions arise for correlations of spins fixed on the boundary of one face, and review the duality relations in this case. In Section 4 we define annular partitions, and relate them to correlations with spins fixed on two boundary components. We also derive some new duality relations for these correlations, and in Section 5 we work out one example in detail. The appendix contains details on some constructions for annular partitions.

2. THE POTTS MODEL AND PARTITIONS

2.1. Review of Potts Model

The partition function for the q-state Potts model on a graph G is

$$Z(G;K,g)=\sum_{\sigma}\prod_{\langle ij\rangle}\exp[(K\delta(\sigma(i)-\sigma(j))] \tag{2}$$

where K is a coupling parameter and δ is the Kronecker delta. Here σ is an assignment of spins on the vertices of G, and $\{\langle ij\rangle\}$ are edges of G.

Let U be a subset of n vertices in G, and let $x=(x_1,\ldots,x_n)$ be an assignment of spin values to the vertices in U. Then the correlation function with fixed spin values x in U is

$$\left\langle\prod_{i\in U}\delta(\sigma(i)-x_i)\right\rangle=Z^{-1}Z_x(G;K,q) \tag{3}$$

where the *partial partition function* is defined by

$$Z_x(G; K, q) = \sum_{\sigma : \sigma(U) = x} \prod_{\langle ij \rangle} \exp[K\delta(\sigma(i) - \sigma(j))] \tag{4}$$

and we have written $\{\sigma(U) = x\}$ to mean $\{\sigma(i) = x_i\}$ for all $i \in U$. From here on we shall focus our attention on this partial partition function rather than on the corresponding correlation function, noting that (3) provides the mapping between them.

2.2. Partitions

Next we recall some basic facts about partitions. Let A be a set with N elements. A *partition* of A is a collection of disjoint subsets of A, called *blocks*, such that every point in A belongs to exactly one block. A partition X is called a *refinement* of the partition Y, if every block of X is a subset of a block of Y. In this case we write $X \leqslant Y$. This defines a partial order on the partitions of A, and the resulting partially ordered set (poset) is denoted $\mathscr{P}$. Suppose that f, g are functions on $\mathscr{P}$, such that for every $Y \in \mathscr{P}$

$$f(Y) = \sum_{X \leqslant Y} g(X) \tag{5}$$

Then it follows that for every $Y \in \mathscr{P}$ we have

$$g(Y) = \sum_{X \leqslant Y} \mu(X, Y) \, f(X) \tag{6}$$

where $\mu(X, Y)$ is the Möbius inversion function.[2] The value of $\mu(X, Y)$ is known; suppose that Y has k blocks $B_1, ..., B_k$, and that B_i contains n_i blocks of X, for $i = 1, ..., k$. Then the Möbius function is given by the formula

$$\mu(X, Y) = \prod_{i=1}^{k} (-1)^{n_i - 1} (n_i - 1)! \tag{7}$$

2.3. Rooted Graph

Following the notation of ref. 4, we will refer to the pair (G, U) as a *rooted graph*. There are q^n possible assignments x for spins in U. Each assignment defines a partition of U, wherein vertices i, j belong to the same block if and only if they are assigned the same spin value. The partial partition function depends only on this partition (this is a special feature of the Potts model). Accordingly for any partition X of U we define

$$Z_X = Z_x(G; K, q) \tag{8}$$

where x is any spin assignment that produces the partition X. We assume henceforth that $q \geqslant n$ so that all partitions of U can be realised by spin assignments.

2.4. Random Cluster Expansion

The random cluster representation of the Potts model partition function is obtained by expanding the interaction term in (2). For each edge $\langle ij \rangle$ we write

$$\exp\left[K\delta(\sigma(i)-\sigma(j))\right] = 1 + (e^K - 1)\,\delta(\sigma(i)-\sigma(j)) \tag{9}$$

and then multiply out the resulting terms over all the edges. This expansion produces a sum over sets of edges in the graph, and each edge set defines a spanning graph of G. Summing over spins enforces the condition that the spins in each connected component of a spanning graph have the same value. We obtain

$$Z(G; K, q) = \sum_{S \subset E} (e^K - 1)^{|S|}\, q^{p(S)} \tag{10}$$

where E is the set of edges on G, $|S|$ denotes the number of edges in the set S, and $p(S)$ is the number of connected components in the spanning graph of S.

We can apply the same expansion to the partial partition function (4). Now each set S appearing in the sum defines a partition of U, which we write $\pi(S)$, by the rule that two vertices are in the same block if and only if they belong to the same connected component of the spanning graph defined by S. Since any set S that appears in the sum from the random cluster expansion of Z_X cannot connect sites in U with different spin assignments, each such set S must induce on U a partition $\pi(S)$ which is a refinement of X. In the language of partially ordered sets, we must have $\pi(S) \leqslant X$. Therefore we get:

$$Z_X = \sum_{S \subset E\,:\,\pi(S) \leqslant X} (e^K - 1)^{|S|}\, q^{p(S) - |\pi(S)|} \tag{11}$$

where $|X|$ denotes the number of blocks in a partition X, and $\pi(S)$ is the partition of U induced by S.

It is useful to isolate the terms in (11) which produce a given partition of U. So for any b, c and partition X we define

$$R_X(G; b, c) = \sum_{S \subset E : \pi(S) = X} b^{|S|} c^{p(S)} \tag{12}$$

Then we have the identity

$$Z_X = \sum_{Y \leqslant X} q^{-|Y|} R_Y(G; e^K - 1, q) \tag{13}$$

Using (6) and (7) this yields

$$R_X(G; e^K - 1, q) = q^{|X|} \sum_{Y \leqslant X} \mu(Y, X)\, Z_Y \tag{14}$$

3. NONCROSSING PARTITIONS

3.1. Definition

As described before, the noncrossing (NC) partitions of $\{1, 2, ..., n\}$ are found by placing the n points in order on the boundary of a disk, and connecting points by arcs across the interior of the disk. A partition is NC if the resulting path-connected components are disjoint. More formally, let U be the set of n distinct points on the unit circle. Note that any two points (u_i, u_j) in U divide the circle into two disjoint arcs. Let (u_k, u_l) be another pair of points in U. Then either both u_k and u_l lie on the same arc defined by (u_i, u_j), or they lie on different arcs. If they lie on *different* arcs we say that the pairs (u_i, u_j) and (u_k, u_l) are *crossing*. A partition X is NC if there are no crossing pairs (u_i, u_j) and (u_k, u_l), where u_i, u_j are in one block and u_k, u_l are in a different block. The number of such partitions is the nth Catalan number $C_n = 1/(n+1)\binom{2n}{n}$.

3.2. Self-Duality of $NC^A(n)$

One property of $NC^A(n)$ will be of importance later, namely the fact that the poset is selfdual.[8, 4] In order to describe the map which implements self-duality, note that the set U has a natural dual set U^* on the circle—these are the n edges connecting adjacent points of U. We can identify each edge with a point midway between two adjacent points of U, and then U^* is also represented by a set of n points on the circle. Self-duality of $\mathrm{NC}^{\mathrm{A}}(n)$ is the observation that every NC partition X of U defines a unique NC partition X^* of U^*, which we call its *dual partition* (note that there is no natural notion of dual for an arbitrary partition). The dual partition is defined as follows: first note that any two points x, y in U^* divide the circle into two disjoint arcs, and hence they produce a partition of U containing exactly two blocks—namely the points of U on the two arcs. Let $P(x, y)$ be this two-block partition of U. Then x, y belong to the same block of the dual partition X^* if and only if X is a refinement of $P(x, y)$. Again there is a simple pictorial representation of this relation—we connect all vertices in each block by arcs, so that arcs of different blocks are disjoint. If x, y can be connected by an arc which does not cross any of these arcs, then x, y are within the same block of X^*.

3.3. Spins Fixed on One Face of Planar Graph

The paper[4] considers the case when G is a connected planar graph, and U lies on the boundary of one face. In this case $R_X(G; b, c) = 0$ unless X is a NC partition, since the allowed sets S are constrained by the topology of the graph. It follows from (13) that the number of independent partial partition functions $\{Z_X\}$ is at most the number of NC partitions, and that if X is *not* NC, then Z_X is a linear combination of the partial partition functions of the NC partitions. These linear relations were derived in ref. 4, and some concrete examples worked out.

3.4. Rooted Planar Graph and Its Dual

We now recall the construction of the dual rooted graph (G^*, U^*) when G is planar, and U lies on one face boundary.[4] Suppose that U contains n points, and lies on the boundary of a face f. We introduce one extra vertex v inside f, and connect v to the vertices of U on the boundary of f by inserting N additional non-overlapping edges inside f, call them $(e_{(1)},..., e_{(N)})$. The resulting graph, call it G_U, is planar and so has a dual $G_U^{(D)}$. Further every edge in G_U has a unique dual edge in $G_U^{(D)}$, and vice versa.

We obtain the graph G^* by removing from $G_U^{(D)}$ the edges which are dual to $(e_{(1)},..., e_{(N)})$. Consequently the graph G^* has N vertices inside the face f, and these constitute the set U^*. If we think of the boundary of f as a circle, then the vertices of U are points on this circle. We can think of the vertices of U^* as also lying on this circle, placed in between consecutive vertices of U. In other words, U^* is the one-dimensional dual of U.

3.5. Duality Relations for Spins Fixed on One Face

The mapping from (G, U) to (G^*, U^*) also provides duality relations for correlation functions.[4, 12] Consider the dual Potts model on the rooted graph (G^*, U^*), whose coupling K^* is related to K by

$$(e^K - 1)(e^{K^*} - 1) = q \tag{15}$$

Then the partial partition functions on (G, U) and (G^*, U^*) are linearly related. That is, for each partition X of U and each partition Y of U^*, there is a number $J(X, Y)$ depending on K, q and $|U|$ such that

$$Z_X = \sum_Y J(X, Y)\, Z^*_Y \tag{16}$$

where Z^*_Y is the partial partition function on (G^*, U^*) with spins assigned to U^* so as to produce the partition Y. We will derive similar results in the next section, for correlations with spins fixed on two boundary components.

4. ANNULAR PARTITIONS

4.1. Definition

We now define the class $\mathrm{NC}^{\mathrm{A}}_2(n, m)$. Let ∂A be the boundary of the annulus

$$A = \{(x, y) : 1 \leqslant x^2 + y^2 \leqslant 2\}, \tag{17}$$

where we choose orientation so that both components of the boundary have positive orientation in the plane. Let U_1 be a set of n distinct points on one boundary component, labelled $\{1, 2,..., n\}$ in order around the circle, and let U_2 be a set of m points on the other boundary component, labeled $\{n+1, 2,..., n+m\}$ in order. Let $U = U_1 \cup U_2$.

We say that X is a (n, m)-*annular* partition of $\{1, 2,..., n+m\}$ if there are arcs in A connecting vertices of U such that (a) each block of X is path connected, and (b) arcs for different blocks do not intersect.

As special cases we note that $\mathrm{NC}^{\mathrm{A}}_2(n, 0) = \mathrm{NC}^{\mathrm{A}}(n)$, and $\mathrm{NC}^{\mathrm{A}}_2(0, m) = \mathrm{NC}^{\mathrm{A}}(m)$. Also we may construct an annular partition by taking a disjoint union of noncrossing partitions on the two circles, and hence (1) holds. Next we describe how any annular partition can be built out of noncrossing partitions on each circle.

4.2. Classification

Given an annular partition Y of U, we define a *bridge* to be any block which contains vertices in both U_1 and U_2. If Y contains k bridges then we say that Y is a *k-bridge* partition. Let $\bar{Y}$ be the zero-bridge partition obtained from Y by splitting each bridge into two blocks, each containing the vertices on one of the circles. Then if Y is a k-bridge, it follows that $|\bar{Y}| = |Y| + k$ (recall that $|X|$ is the number of blocks in a partition X). Reversing the process, every k-bridge with $k \geqslant 1$ can be obtained by starting with a zero-bridge and joining together blocks on the two different faces. However there are restrictions on which blocks can be joined from the two faces. Namely it must be possible to draw arcs connecting all vertices in each block, so that the arcs of different blocks do not intersect. This means in particular that the blocks cannot be nested inside each other. We give a precise definition of this construction in the appendix.

4.3. Partial Duality

There is also a notion of partial duality for annular partitions, which we now explain. Each of the sets U_1 and U_2 has a one-dimensional dual, namely U_1^* and U_2^* respectively, constructed in the way described before. We define U^* to be the union of U_1^* and U_2^*. We call our duality "partial" because it does not provide a 1–1 mapping between annular partitions of U and U^*.

If X is a k-bridge partition with $k \geqslant 2$, then X also has a unique and well-defined dual X^*. The easiest way to see this is by drawing a picture—see Fig. 1 for an example of a partition with $k = 2$. The vertices in each

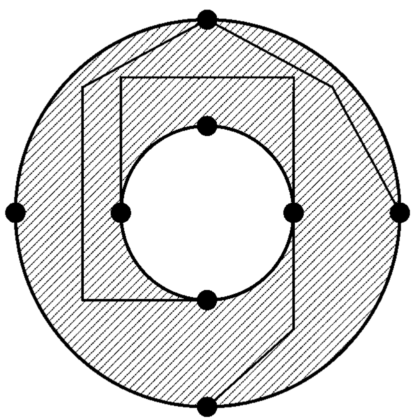

Fig. 1. Example of a two bridge partition X.

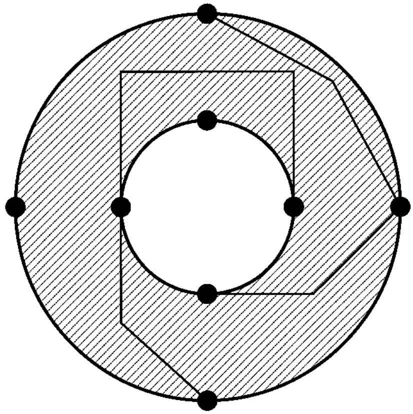

Fig. 2. Same partition X, different edge set.

bridge are connected by arcs which cross the annulus between the two boundary components. The remaining blocks on each boundary component are connected by nonintersecting arcs. Together these define a partition of U^*, by the rule that two vertices are in the same block if there is an arc between them which does not cross any of these arcs. For $k \geqslant 2$ this partition does not depend on any choices made when drawing the arcs, see for example Figs. 1, 2 and 3, so the dual X^* is well defined. It is also possible to turn this pictorial explanation into a formal definition of X^*, and for completeness we do this in the appendix.

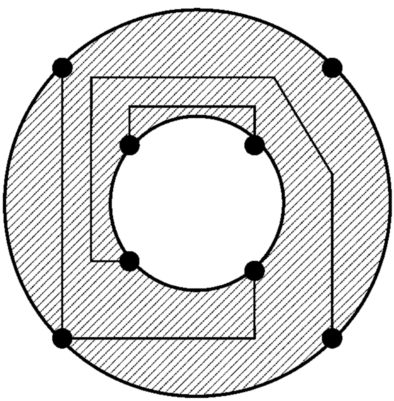

Fig. 3. Dual partition X^*.

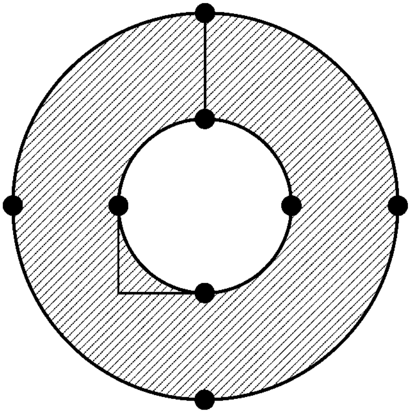

Fig. 4. Example of a one bridge partition Y.

Now suppose that X is a zero-bridge partition of U; then X is composed of two NC partitions, one each for U_1 and U_2. Each of these NC partitions has a unique dual NC partition (constructed as in Section 3.2), and together they form a zero-bridge partition of U^*. So for a zero-bridge partition we make this our *definition* of the dual partition X^*.

However if $k = 1$, that is if X is a one-bridge partition, the partition of U^* obtained by the above construction does depend on the choices made, see for example Figs. 4 and 5. Hence there does not seem to be any useful notion of dual partition for a one-bridge.

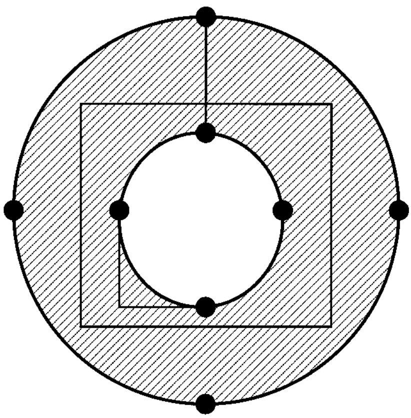

Fig. 5. Same partition Y, but *different* partition of U^*.

4.4. Random Cluster Expansion

As discussed earlier, the random cluster expansion of the Potts model on (G, U) produces a sum over annular partitions. Specifically every edge set S defines an annular partition $\pi(S)$ of U, and in addition S provides the connecting arcs which make each block path-connected.

4.5. The Dual Rooted Graph

Now we explain the construction of the dual rooted graph (G^*, U^*) when U contains vertices lying on the boundaries of two disjoint faces in G. It proceeds in parallel with the construction in Section 3.4. Without loss of generality we take one of the faces to be unbounded. Let f_1 and f_2 be the two faces, and suppose there are n and m vertices of U on the boundaries of f_1 and f_2 respectively. We introduce one extra vertex v_1 inside f_1, and one extra vertex v_2 inside f_2. We then connect v_1 to the vertices of U on the boundary of f_1 by inserting n additional non-overlapping edges inside f_1, call them $(e_{(1,1)},..., e_{(1,n)})$. Similar edges are added inside f_2, call them $(e_{(2,1)},..., e_{(2,m)})$. The resulting graph G_U is planar and so has a dual $G_U^{(D)}$. Further every edge in G_U has a unique dual edge in $G_U^{(D)}$, and vice versa.

We obtain the graph G^* by removing from $G_U^{(D)}$ the edges which are dual to $(e_{(1,1)},..., e_{(1,n)})$ and $(e_{(2,1)},..., e_{(2,m)})$. The graph G^* has n vertices inside the face f_1 and m vertices in the face f_2, and together these form the set U^*. The pair (G^*, U^*) is the dual rooted graph.

4.6. Duality Relations

Given an edge set S on G, we define the dual set S^* on G^* by the condition that an edge is in S^* if and only if its dual edge is not in S. As usual we denote by $\pi(S)$ the annular partition of U induced by S. Similarly S^* defines an annular partition of U^*, namely $\pi(S^*)$.

If $\pi(S)$ is a k-bridge partition with $k \geqslant 2$, then $\pi(S^*)$ is also a k-bridge partition, and it is precisely the dual partition defined in Section 4.3. However if $k=0$ or $k=1$ then $\pi(S^*)$ does *not* depend alone on $\pi(S)$. In general it also depends on S. This is consistent with the fact that there is no natural notion of duality for zero-bridge and one-bridge partitions. One consequence of this is that there is no natural 1-1 mapping between partial partition functions on (G, U) and (G^*, U^*), as was the case when U belonged to one face boundary.

In order to salvage the situation, we divide the zero-bridge and one-bridge partitions into classes, as follows. For each zero-bridge partition X, let $\mathscr{C}(X)$ be the set of annular partitions consisting of X together with all one-bridge partitions which can be constructed from X by joining blocks on the two faces. Now X itself has a dual X^*, namely the zero-bridge partition formed by taking the planar duals of its two planar components. We have the following result.

Lemma 1. Let X be a zero-bridge partition, and S an edge set. Then $\pi(S)$ is in the class $\mathscr{C}(X)$ if and only if $\pi(S^*)$ is in the class $\mathscr{C}(X^*)$.

Proof. If an edge set S contains a closed path that seperates the two boundary components, then the dual edge set S^* cannot connect the two dual boundary components. Hence $\pi(S^*)$ must be a zero-bridge in this case. By examining the cases when both, only one of, or neither S and S^* contain such closed paths, the result can be deduced.

Recall that for any partition X we defined R_X in (12) using the sum over all edge sets S such that $\pi(S) = X$. Now in addition we define $\chi(S)$ to be zero if $\pi(S)$ is a zero-bridge partition, and equal to one otherwise. Then there is a simple relation between the edge set S and its dual S^*, as follows.

Lemma 2. For all b, c

$$b^{|S|}c^{p(S)} = b^{|E|}c^{3-|F|-|\pi(S^*)|-\chi(S^*)}(b^{-1}c)^{|S^*|}c^{p(S^*)} \tag{18}$$

Proof. First observe that

$$|S| + |S^*| = |E| \tag{19}$$

where $|E|$ is the number of edges in G. Second,

$$p(S) = p(S^*) + |\pi(S)| + |S^*| + \chi(S) - |F| - N + 1 \tag{20}$$

where $|F|$ is the number of faces in G, and $N = |U| = n + m$ is the number of roots. Third,

$$|\pi(S)| + |\pi(S^*)| = N + 2 - \chi(S) - \chi(S^*) \tag{21}$$

4.7. Partial Duality for Partial Partition Functions

By combining Lemma 1 and Lemma 2, we get the following set of relations. First, if X is a k-bridge partition with $k \geqslant 2$ then

$$R_X(G; b, c) = b^{|E|}c^{2-|F|-|X^*|}R_{X^*}(G^*; b^{-1}c, c) \tag{22}$$

Second, if X is-a zero-bridge partition then

$$\sum_{Y \in \mathscr{C}(X)} R_Y(G; b, c) = b^{|E|} c^{3-|F|-|X^*|} \sum_{W \in \mathscr{C}(X^*)} R_W(G^*; b^{-1}c, c) \tag{23}$$

Using the Möbius inversion formula, both (22) and (23) provide linear relations between the partial partition functions on G and G^*. By substituting (14) and the corresponding relation on G^* into (22) and (23) and taking $b = e^K - 1$, $c = q$ we obtain the following linear duality relations for partial partition functions, and hence for correlations. These are the analogs of the duality relations found in ref. 4. In the next section we work out the relations in full detail for one example.

Case 1. X is a k-bridge with $k \geqslant 2$.

$$q^{|X|} \sum_{Y \leqslant X} \mu(Y, X)\, Z_Y = (e^K - 1)^{|E|}\, q^{2-|F|} \sum_{W \leqslant X^*} \mu(W, X^*)\, Z^*_W \tag{24}$$

Case 2. X is a 0-bridge.

$$\sum_{V \in \mathscr{C}(X)} q^{|V|} \sum_{Y \leqslant V} \mu(Y, V)\, Z_V$$
$$= (e^K - 1)^{|E|}\, q^{3-|F|-|X^*|} \sum_{R \in \mathscr{C}(X^*)} q^{|R|} \sum_{W \leqslant R} \mu(W, R)\, Z^*_R \tag{25}$$

Note that there are not enough relations to allow a complete solution for Z_X in terms of Z^*_Y. The number of relations depends on the numbers of vertices of U on both faces. It is equal to the number of annular partitions minus the number of one-bridges. This is strictly less than the number of independent partial partition functions, and we conjecture that (24) and (25) constitute a complete set of independent relations between the Z_X and Z^*_Y.

Note that the number of one-bridges can be calculated in terms of the NC partitions. Let $\mathscr{P}_1$ and $\mathscr{P}_2$ be the sets of all NC partitions on the two faces. Then the number of one-bridges is

$$\left(\sum_{X \in \mathscr{P}_1} |X|\right)\left(\sum_{X \in \mathscr{P}_2} |X|\right) \tag{26}$$

5. EXAMPLE OF DUALITY RELATIONS

We work out in full detail the duality relations (24) and (25) for a special case, where there are four fixed spins in U, with two each on two

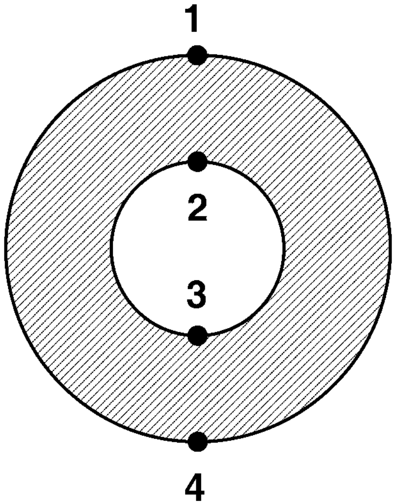

Fig. 6. Example with two roots on each face.

disjoint faces. The setup is shown in Fig. 6, where the vertices in U are labeled $\{1, 2, 3, 4\}$ (without loss of generality we have chosen one of the faces as the infinite face of the graph). The vertices in U^* are labeled $\{5, 6, 7, 8\}$, as shown in Fig. 7.

There are 15 possible partitions of U, and all of them are annular partitions. There are four 0-bridge partitions, nine 1-bridge partitions and two 2-bridge partitions—they are listed in Table I. For example the partition (12)(3)(4) has three blocks, and one of these (12) is a bridge.

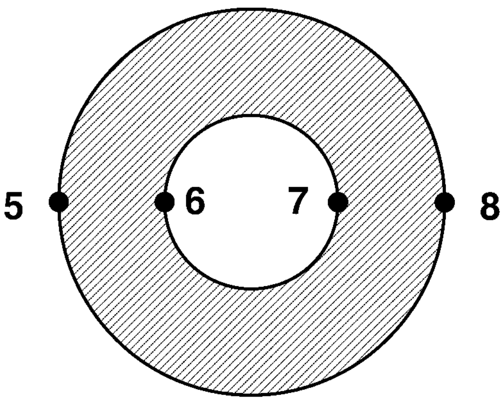

Fig. 7. Dual of previous example.

Table I. List of All Annular Partitions for Rooted Graphs (G, U) and (G^*, U^*) in Figs. 6 and 7

Number of bridges	Partitions of U	Partitions of U^*
0	(1)(2)(3)(4)	(5)(6)(7)(8)
	(14)(2)(3)	(58)(6)(7)
	(1)(23)(4)	(5)(67)(8)
	(14)(23)	(58)(67)
1	(12)(3)(4)	(56)(7)(8)
	(13)(2)(4)	(57)(6)(8)
	(1)(24)(3)	(5)(68)(7)
	(1)(2)(34)	(5)(6)(78)
	(123)(4)	(567)(8)
	(124)(3)	(568)(7)
	(134)(2)	(578)(6)
	(234)(1)	(678)(5)
	(1234)	(5678)
2	(12)(34)	(56)(78)
	(13)(24)	(57)(68)

Hence there are six duality relations between $R_X(G)$ and $R_Y(G^*)$, and we list them below. They imply six relations between Z_X and Z^*_Y which we also list.

Relations for R. The relations hold for general arguments b, c of the functions R_X. We define

$$\lambda = b^{|E|} c^{-|F|} \tag{27}$$

Let ϕ: $U \to U^*$ be the map $1 \to 5$, $2 \to 6$, $3 \to 7$ and $4 \to 8$. To keep the notation as compact as possible, we will use a partition X of U to also denote the corresponding partition $\phi(X)$ of U^*. Then we will write $T_X = \lambda^{-1/2} R_X(G; b, c)$ and also $T^*_X = \lambda^{1/2} R_{\phi(X)}(G^*; b^{-1}c, c)$ so for example $T^*_{(12)(3)(4)} = \lambda^{1/2} R_{(56)(7)(8)}(G^*; b^{-1}c, c)$.

We list four relations below between the T_X and the T^*_Y. The other two relations are obtained by exchanging T and T^* in the first two relations.

$$T_{(1)(2)(3)(4)} + T_{(12)(3)(4)} + T_{(13)(24)} + T_{(1)(3)(24)} + T_{(1)(2)(34)} = c[T^*_{(14)(23)} + T^*_{(1234)}] \tag{28}$$

$$T_{(14)(2)(3)} + T_{(124)(3)} + T_{(134)(2)} = T^*_{(1)(4)(23)} + T^*_{(123)(4)} + T^*_{(1)(234)} \tag{29}$$

$$T_{(12)(34)} = T^*_{(12)(34)} \tag{30}$$

$$T_{(13)(24)} = T^*_{(13)(24)} \tag{31}$$

Relations for Z. We set $c=q$ and $b=e^K-1$, and define for any partition X of U

$$W_X = \sum_{Y \leqslant X} q^{-|Y|} T_Y$$

$$W_X^* = \sum_{Y \leqslant X} q^{-|Y|} T_Y^*$$

Then $Z_X = \lambda^{1/2} W_X$ and $Z^*_{\phi(X)} = \lambda^{-1/2} W_X^*$, where now $\lambda = (e^K - 1)^{|E|}\, q^{-|F|}$.

Again we list four relations—the other two are obtained by exchanging W_X and W_X^* in the first two relations.

$$\begin{aligned} &W_{(12)(3)(4)} + W_{(13)(2)(4)} + W_{(1)(3)(24)} + W_{(1)(2)(34)} - 3W_{(1)(2)(3)(4)} \\ &\quad = q[\,W^*_{(1234)} - W^*_{(123)(4)} - W^*_{(124)(3)} - W^*_{(134)(2)} - W^*_{(234)(1)} - W^*_{(12)(34)} \\ &\qquad - W^*_{(13)(24)} + 2W^*_{(12)(3)(4)} + 2W^*_{(13)(2)(4)} + W^*_{(14)(2)(3)} + W^*_{(1)(4)(23)} \\ &\qquad + 2W^*_{(24)(1)(3)} + 2W^*_{(1)(2)(34)} - 5W^*_{(1)(2)(3)(4)}] \end{aligned} \tag{32}$$

$$\begin{aligned} &W_{(124)(3)} + W_{(134)(2)} - W_{(12)(3)(4)} - W_{(24)(1)(3)} - W_{(13)(2)(4)} \\ &\qquad - W_{(14)(2)(3)} - W_{(34)(1)(2)} + 3W_{(1)(2)(3)(4)} \\ &\quad = W^*_{(123)(4)} + W^*_{(1)(234)} - W^*_{(12)(3)(4)} - W^*_{(13)(2)(4)} - W^*_{(1)(2)(34)} \\ &\qquad - W^*_{(1)(3)(24)} - W^*_{(1)(4)(23)} + 3W^*_{(1)(2)(3)(4)} \end{aligned} \tag{33}$$

$$\begin{aligned} &W_{(12)(34)} - W_{(12)(3)(4)} - W_{(1)(2)(34)} + W_{(1)(2)(3)(4)} \\ &\quad = W^*_{(12)(34)} - W^*_{(12)(3)(4)} - W^*_{(1)(2)(34)} + W_{(1)(2)(3)(4)} \end{aligned} \tag{34}$$

$$\begin{aligned} &W_{(13)(24)} - W_{(13)(2)(4)} - W_{(1)(3)(24)} + W_{(1)(2)(3)(4)} \\ &\quad = W^*_{(13)(24)} - W^*_{(13)(2)(4)} - W^*_{(1)(3)(24)} + W^*_{(1)(2)(3)(4)} \end{aligned} \tag{35}$$

APPENDIX. ANNULAR PARTITIONS AND THEIR DUALS

Making *k*-Bridges

First we describe how all annular partitions are constructed from zero-bridge partitions. As usual, U contains $n+m$ points on two circles forming the boundary of an annulus. Given two points x, y on one of these circles, let $A_1(x, y)$ and $A_2(x, y)$ be the two arcs of the circle between x and y. Let

$\mathscr{B} = \{B_1,..., B_k\}$ be a collection of disjoint subsets of U lying on this circle (since it need not include all vertices, it is in general not a partition). Each set B_i divides the boundary into $|B_i|$ disjoint arcs, call them $\{S_i, a\}$, $a = 1,..., |B_i|$. We say that $\mathscr{B}$ is a *k-chain* if for each set B_i, there is one arc $S_{i,a}$ which contains all other sets B_j in $\mathscr{B}$. In this case we call $S_{i,a}$ the *exterior arc* of B_i. Note that if $k = 1$ then $\mathscr{B}$ is always a 1-chain, but the exterior arc is not defined.

Now suppose that X is a zero-bridge partition, and suppose also that it contains a k-chain on both circles. Then we can form a k-bridge annular partition by joining the blocks in the k-chains together. There are exactly k different k-bridges that can be formed from this pair of k-chains, and each is specified uniquely by giving the component blocks of any one bridge. This gives a complete characterization of annular partitions.

Making the Dual

Next we give a formal definition of the dual of a k-bridge partition X when $k \geqslant 2$. To do this we have to give a condition which determines whether two dual points x, y in U^* lie in the same block of X^*. First suppose that the dual points x, y lie on the same boundary circle. Let $P(x, y)$ be the two-block partition of points on that boundary defined by x, y. Recall that $\bar{X}$ is obtained from X by splitting each bridge into two blocks, one for each boundary component. Let X_1 be the restriction of X to the boundary component containing x, y. Then x, y are in the same block of X^* if and only if X_1 is a refinement of $P(x, y)$, and all blocks of X_1 which come from bridges of X lie in the *same* block of $P(x, y)$.

Next recall that X is built using a k-chain on both circles. For $k \geqslant 2$, each block in a k-chain has an exterior arc which contains all other blocks in the k-chain. Clearly these arcs overlap for every pair of blocks in the k-chain. Let S_1 and S_2 be the intersections of these exterior arcs on both circles. Then each S_i is a union of k disjoint arcs. Each of these arcs has endpoints belonging to different bridges in the k-chain. Say the arc s_1 in S_1 has endpoints on the bridges B_i and B_j. Then there is a unique arc s_2 in S_2 whose endpoints also lie on B_i and B_j. In this case we call s_1 and s_2 the *matching arcs* in S_1 and S_2. This gives a 1-1 correspondence between S_1 and S_2.

Now suppose that x, y lie on different face boundaries. Then x, y are in the same block of X^* if and only if (i) they lie on matching arcs s_1 in S_1 and s_2 in S_2, (ii) in s_1 all blocks of X lie wholly in either $A_1(x)$ or $A_2(x)$, and (iii) in s_2 all blocks of X lie wholly in either $A_1(y)$ or $A_2(y)$.

Note that for $k = 1$ this construction is not well defined since there is no exterior arc.

ACKNOWLEDGMENT

The author thanks F. Y. Wu for valuable discussions and insights. Research supported in part by NSF Grant DMS–9705779.

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